数理解析研究所講究録
第 1712 巻 2010 年 58-63

The Parents and the Children of Non-Weierstrass semigroups [1]

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Abstract

For a numerical semigroup H of genus g Bras-Amorós [1] defined a numerical semigroup $p(H)$ of genus $g-1$ which is called the *parent* of H. She also called H a *child* of $p(H)$. We consider three kinds of non-Weiersrass semigroups. Children of some non-Weierstrass semigroups in each case are investigated.

1 Parents and children of numerical semigroups

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. Let $f(H)$ be the greatest number of $\mathbb{N}_0\backslash H$, which is called the *Frobenius number* of H.

Example 1.1 Let $H=\langle 3,5,7\rangle$. Then we have $\mathbb{N}_0\backslash H=\{1,2,4\}$, which implies that $g(\langle 3,5,7\rangle)=3$. Moreover, we get $f(\langle 3,5,7\rangle)=4$.

Let $p(H)=H\cup\{f(H)\}$, which is called the *parent* of H. Then $p(H)$ is a numerical semigroup of genus $g(H)-1$. Moreover, H is called a *child* of $p(H)$. These definitions are due to Bras-Amorós [1].

Example 1.2 We have $p(\langle 3,5,7\rangle)=\langle 3,4,5\rangle$. Hence, the semigroup $\langle 3,4,5\rangle$ is the parent of $\langle 3,5,7\rangle$. Conversely, the semigroup $\langle 3,5,7\rangle$ is a child of $\langle 3,4,5\rangle$.

Let $M(H)$ be a unique minimal set of generators for H. An element $\mu\in M(H)$ is called an *effective minimal generator* if $\mu>f(H)$. When

[1] This paper is an extended abstract and the details will appear elsewhere.

$\{\mu_1 < \ldots < \mu_{r+e}\} = M(H)$ where μ_i's ($i \geqq r+1$) are effective, we denote H by $\langle \mu_1, \ldots, \mu_r \mid \mu_{r+1}, \ldots, \mu_{r+e} \rangle$.

Remark 1.1 *For an effective minimal generator μ we set $H_\mu = H \backslash \{\mu\}$. Then H_μ becomes a numerical semigroup of genus $g(H)+1$. Moreover, the set $p^{-1}(H)$ of children of H is $\{H_\mu \mid \mu$ is an effective minimal generator of $H\}$.*

Example 1.3 i) Let $H = \langle |2, 3\rangle$. Then $H_3 = \langle 2|5\rangle$ and $H_2 = \langle |3, 4, 5\rangle$ are the children of H.
ii) Let $H = \langle |3, 4, 5\rangle$. Then the children of H are $H_5 = \langle 3, 4|\rangle$, $H_4 = \langle 3|5, 7\rangle$ and $H_3 = \langle |4, 5, 6, 7\rangle$.

Example 1.4 $\langle 3, 4|\rangle$ and $\langle 3, 5|\rangle$ have no child.

2 Non-Weierstrass semigroups

In this paper a *curve* means a projective non-singular curve over an algebraically closed field k of characteristic 0. Let $k(C)$ be the field of rational functions on C. A numerical semigroup H is said to be *Weierstrass* if there is a pointed curve (C, P) such that

$$H = H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ with } (f)_\infty = nP\}.$$

Remark 2.1 (Buchweitz [3]) *For $m \geqq 2$, we define*

$$L_m(H) := \{a_1 + \cdots + a_m \mid a_i \in \mathbb{N}_0 \backslash H\}.$$

If there exists m such that $\sharp L_m(H) \geqq (2m-1)(g(H)-1)+1$, then H is non-Weierstrass.

Example 2.1 (Buchweitz [3]) Let $B_{16} = \langle 13 \longrightarrow 18, 20, 22, 23\rangle$ where for two integers $a < b$ the symbol $a \longrightarrow b$ means the consecutive numbers $a, a+1, \ldots, b-1, b$. Then $\mathbb{N}_0 \backslash B_{16} = \{1 \longrightarrow 12, 19, 21, 24, 25\}$, which implies that $g(B_{16}) = 16$ and $\sharp L_2(B_{16}) = 46 = (2*2-1)(16-1)+1$. Hence, B_{16} is non-Weierstrass. We note that $B_{16} = \langle 13 \longrightarrow 18, 20, 22, 23|\rangle$ has no child.

A numerical semigroup H is said to be *Buchweitz* if there exists $m \geqq 2$ such that $\sharp L_m(H) \geqq (2m-1)(g(H)-1)+1$.

Problem 2. 1 Are there non-Weierstrass semigroups which are not Buchweitz?

The above problem was solved by Stöhr and Torres [5]. We will describe their method. For a numerical semigroup H we set

$$d_2(H) = \{\frac{h}{2} \mid h \in H \text{ is even}\},$$

which is also a numerical semigroup.

Remark 2.2 (Stöhr-Torres) *Let H be a numerical semigroup such that $d_2(H)$ is non-Weierstrass. If $g(H) \geqq 6g(d_2(H)) + 4$, then H is non-Weierstrass.*

Example 2. 2 (Stöhr-Torres) Let B_{16} be as in Example 2.1. Assume that $g \geqq 100 = 6 * 16 + 4$. We set

$$ST_g = 2B_{16} \cup \{2g - 1 - 2t \mid t \in \mathbb{Z} \backslash B_{16}\}.$$

Then we have $d_2(ST_g) = B_{16}$ and $g(ST_g) = g$. By Remark 2.2 we see that ST_g is a non-Weierstrass semigroup, which is non-Buchweitz. Moreover, we obtain

$$ST_{100} = 2\langle 13 \longrightarrow 18, 20, 22, 23\rangle + \langle 149, 151, 157, 161|\rangle$$

with $f(ST_{100}) = 199$. Hence, ST_{100} has no child.

A non-Buchweitz numerical semigroup H is said to be *Stöhr-Torres* if there exists a finite sequence $\{H_i\}_{i=0,1,\dots,n}$ with Buchweitz H_0, $d_2(H_i) = H_{i-1}$ and $H_n = H$ satisfying $g(H_i) \geqq 6g(H_{i-1}) + 4$ for $i = 1, \dots, n$. Here, we pose the following problem:

Problem 2. 2 Are there non-Weierstrass semigroups which are neither Buchweitz nor Stöhr-Torres?

The above problem was also solved in [4]. In fact, we have the following:

Example 2. 3 ([4]) $H = \langle 8, 12, 18, 22, 51, 55\rangle$ is a non-Weierstrass semigroup which is neither Buchweitz nor Stöhr-Torres. In fact, the numerical semiroup $d_2(H) = \langle 4, 6, 9, 11\rangle$ of genus 5 is Weierstrass. Moreover, we get $g(H) = 34$ and $f(H) = 69$. Hence, we obtain $H = \langle 8, 12, 18, 22, 51, 55|\rangle$, which implies that H has no child.

From the results in this section we have the following problem which will be considered in the next section:

Problem 2. 3 Are there non-Weierstrass semigroups which have children?

3 Infinite chains

We can find a Buchweitz semigroup which has a child.

Example 3. 1 We set

$$H = \langle 112 \longrightarrow 119, 121 \longrightarrow 143, 145 \longrightarrow 215, 217, 218, 219 | 223 \rangle.$$

The semigroup H is Buchweitz. It has a chilld $H' = H \backslash \{223\}$ which is also Buchweitz. In this case, $f(H) = 222$ and $g(H) = 117$.

We also see that there is a Stöhr-Torres semigroup with a child.

Example 3. 2 We set

$$H = 2\langle 13 \longrightarrow 18, 20, 22, 23 \rangle + \langle 151, 153, 169, 163 | 201 \rangle.$$

The semigroup H with $f(H) = 175$ is Stöhr-Torres. It has a chilld $H' = H \backslash \{201\} = ST_{101}$ which is also Stöhr-Torres. In this case, we have $g(H) = 100$ and $d_2(H) = \langle 13 \longrightarrow 18, 20, 22, 23 \rangle = B_{16}$.

We are interested in numerical semigroups which leave offsprings through all eternity. So, we define the following: A numerical semigroup H has an *infinite chain* if there exists an infinite sequence $\{H^{(i)}\}_{i \geqq 0}$ such that $H^{(0)} = H$ and $p(H^{(i)}) = H^{(i-1)}$ for all $i \geqq 1$. This definition is due to Bras-Amorós and Bulygin [2].

Example 3. 3 i) Let $g \geqq 1$. We set $H = \langle 2, 2g+1 \rangle$. Then H has an infinite chain. In fact, we get

$$p(\langle 2, 2(g+i)+1 \rangle) = \langle 2, 2(g+i-1)+1 \rangle$$

for all $i \geqq 1$. Namely we have an infinite sequence $\{H^{(i)} = \langle 2, 2(g+i)+1 \rangle\}_{i \geqq 0}$ with $H^{(0)} = H$.

ii) Let $g \geqq 1$. We set $H = \langle g+1 \longrightarrow 2g+1 \rangle$, which is ordinary. Then H has an infinite chain, because we obtain

$$p(\langle g+i+1 \longrightarrow 2(g+i)+1 \rangle) = \langle g+(i-1)+1 \longrightarrow 2(g+i-1)+1 \rangle$$

for all $i \geqq 1$. That is to say, we have an infinite sequence

$$\{H^{(i)} = \langle g+i+1 \longrightarrow 2(g+i)+1\rangle\}_{i\geqq 0}$$

with $H^{(0)} = H$.

By the above examples the most general or the most special numerical semigroup has an infinite chain. Morover, we want to consider the following problem:

Problem 3.1 Are there non-Weierstrass semigroups which have infinite chains?

We can find Stöhr-Torres semigroups which have infinite chains.

Theorem 3.1 *Let $n \geqq 10g(B_{16})+9 = 169$ be an odd number. We set*

$$H = 2B_{16} + \langle n, n+2, \ldots, n+2\cdot 15\rangle.$$

Then H is a non-Weierstrass semigroup which has an infinite chain. It is Stöhr-Torres.

Proof. We have

$$p(2B_{16} + \langle n+2i, n+2i+2, \ldots, n+2i+2\cdot 15\rangle)$$
$$= 2B_{16} + \langle n+2(i-1), n+2(i-1)+2, \ldots, n+2(i-1)+2\cdot 15\rangle).$$

Namely we have an infinite sequence

$$\{H^{(i)} = 2B_{16} + \langle n+2i, n+2i+2, \ldots, n+2i+2\cdot 15\rangle\}_{i\geqq 0}$$

with Stöhr-Torres $H^{(i)}$'s. □

Finally, the unsolved problems are presented using the following table:

Problem 3.2 i) Is there a non-Weierstrass semigroup with a child which is neither Buchweitz nor Stöhr-Torres?
ii) Is there a Buchweitz semigroup with an infinite chain?
iii) Is there a non-Weierstrass semigroup with an infinite chain which is neither Buchweitz nor Stöhr-Torres?
Summarizing the results in this paper and the above open problems we get the table.

Properties	Weierstrass	New Type	Stöhr-Torres	Buchweitz
Children	$\exists$	?	$\exists$	$\exists$
Infinite chains	$\exists$	?	$\exists$	?

Here, "New Type" means a non-Weierstrass semigroup which is neither Buchweitz nor Stöhr-Torres. We note that Example 2.3 gives a "New Type" semigroup.

References

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